II.4-DWOPER DYNAMIC WAVE ROUTING

Introduction

DWOPER routing is a dynamic wave flood routing model that routes an inflow hydrograph to a point downstream. It can be used on a single river or system of rivers where storage routing methods are inadequate due to the effects of backwater, tides and mild channel bottom slopes. The model is based on the complete one-dimensional St. Venant equations. A weighted four-point nonlinear implicit finite difference scheme is used to obtain solutions to the St. Venant equations using a Newton-Raphson iterative technique.

DWOPER has a number of features (Fread, 1978) which make it applicable to a variety of natural river systems for real-time forecasting. It is designed to accommodate various boundary conditions and irregular cross sections located at unequal distances along a single multiple-reach river or several such rivers having a dendritic configuration. It allows for roughness parameters to vary with location and stage or discharge. Temporally varying lateral inflows, wind effects, bridge effects, off-channel storage and weir-flow channel bifurcations to simulate levee overtopping are included among its features. Time steps are chosen solely on the basis of desired accuracy since the implicit finite difference technique is not restricted to the very small time steps of explicit techniques due to numerical stability considerations. This enables DWOPER to be very efficient as to computational time for simulating slowly varying floods of several days duration.

Mathematical Basis

The basis for DWOPER is a finite difference solution of the conservation form of the one-dimensional equations of unsteady flow consisting of the conservation of mass and momentum equations:

$$\frac{\partial Q}{\partial x} + \frac{\partial (A+A_o)}{\partial t} - q = 0 \qquad (1)$$

$$\frac{\partial Q}{\partial t} + \frac{\partial (Q^2/A)}{\partial x} + gA\left[\frac{\partial h}{\partial x} + S_f + S_e\right] - qv_x + W_f B = 0 \qquad (2)$$

in which Q is discharge, A is cross-sectional area, A_o is off-channel cross-sectional area wherein flow velocity is considered negligible, h is the water surface elevation, q is lateral inflow or outflow, x is distance along the channel, t is time, g is the gravity acceleration constant, v_x is the velocity of lateral inflow in the x-direction, W_f is the wind term, B is the channel topwidth and S_f is the friction slope defined as:

$$S_f = \frac{n^2|Q|Q}{2.2\ A^2 R^{4/3}} \tag{3}$$

in which n is the Manning's roughness coefficient and R is the hydraulic radius. The term S_e is defined as:

$$S_e = \frac{K_e \partial (Q/A)^2}{2_g\ \partial x} \tag{4}$$

in which K_e is the expansion-contraction coefficient.

Equations 1 and 2 are nonlinear partial differential equations which may be solved by finite difference techniques of explicit or implicit variety. Explicit methods, although simpler in application are not suitable for application of the equations to long-term unsteady flow phenomena such as flood waves because they are restricted by mathematical stability considerations to very small computational time steps (on the order of a few minutes); this causes the explicit techniques to be very inefficient in the use of computer time. Implicit finite difference techniques, however, have no restrictions on time step size other than accuracy considerations.

The 'weighted four-point' implicit scheme is chosen to be the most advantageous of the various implicit schemes which have been proposed from time to time because it can readily be used with unequal distance steps and its stability-convergence properties can be controlled. In the weighted four-point scheme, the continuous x-t region in which solutions of h and Q are sought is represented by a rectangular net of discrete points, as shown in Figure 1 at equal or unequal intervals of Δx and Δt along the x and t axes, respectively.

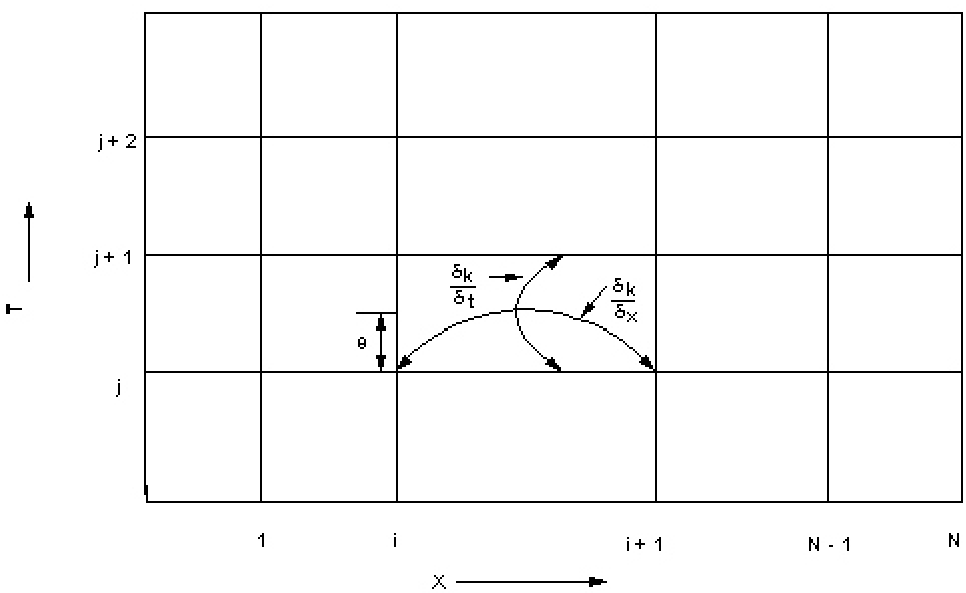

Fig. 1 - Discrete X-T Solution Domain

Each point is identified by a subscript (i) which designates the x position and a superscript (j) for time position. The time derivatives are approximated by:

$$\frac{\partial K}{\partial t} = \left[K_i^{j+1} + K_{i+1}^{j+1} - K_i^{j} - K_{i+1}^{j} \right] / 2\Delta t \qquad (5)$$

in which K represents any variable. The spatial derivatives are approximated by a finite difference quotient positioned between two adjacent time lines according to weighting factors Θ and $(1-\Theta)$, i.e.,

$$\frac{\partial K}{\partial x} = \theta \left[K_{i+1}^{j+1} - K_i^{j+1} \right] \Delta x + (1-\theta) \left[K_{i+1}^{j} - K_i^{j} \right] / \Delta x \qquad (6)$$

and variables other than derivatives are approximated in a similar manner, i.e.,

$$K = \theta \left[K_i^{j+1} + K_{i+1}^{j+1} \right] / 2 + (1-\theta) \left[K_i^{j} + K_{i+1}^{j} \right] / 2 \qquad (7)$$

When θ equals 1.0, a fully implicit scheme is formed. A box scheme results if θ is fixed at 0.5. The influence of the weighting factor on the stability and convergence properties was examined by Fread (1974), who concluded that the accuracy decreases as θ departs from 0.5 and approaches 1.0. This effect becomes more pronounced as the time step size increases. DWOPER allows θ to be an input parameter. A value of 0.55 is often used to minimize loss of accuracy while avoiding weak or pseudo instability when θ of 0.5 is used.

Substitution of the finite difference quotients defined by Equations 5, 6 and 7 into Equations 1 and 2 for the derivatives and non-derivative terms yields the following difference equations:

$$\begin{aligned} &\theta \left(Q_{i+1}^{j+1} - Q_i^{j+1} - q_i^{j+1} \Delta x_1 \right) + (1-\theta) \left(Q_{i+1}^{j} - Q_i^{j} - q_i^{j} \Delta x_i \right) + \\ &0.5 \Delta x_i / \Delta t_j \left[(A+A_o)_i^{j+1} + (A+A_o)_{i+1}^{j+1} - (A+A_o)_i^{j} - (A+A_o)_{i+1}^{j} \right] = 0 \end{aligned} \qquad (8)$$

$$\begin{aligned} &0.5\, \Delta x_i / \Delta t_j \left(Q_i^{j+1} + Q_{i+1}^{j+1} - Q_i^{j} - Q_{i+1}^{j} \right) + \theta \Bigg[\left(Q^2/A\right)_{i+1}^{j+1} - \left(Q^2/A\right)_i^{j+1} \\ &+ g\bar{A}_i^{j+1} \left(h_{i+1}^{j+1} - h_i^{j+1} + \bar{S}_{f_i}^{j+1} \Delta x_i + \bar{S}_{e_i}^{j+1} \Delta x_i \right) - \Delta x_i \left(\bar{q}\ \bar{v}_x \right)_i^{j+1} + \Delta x_i \left(\bar{W}_f \bar{B} \right)_i^{j+1} \Bigg] \\ &+ (1-\theta) \Bigg[\left(Q^2/A\right)_{i+1}^{j} - \left(Q^2/A\right)_i^{j} + g\bar{A}_i^{j} \left(h_{i+1}^{j} - h_i^{j} + \bar{S}_{f_i}^{j} \Delta x_i + \bar{S}_{e_i} \Delta x_i \right) \\ &- \Delta x_i \left(\bar{q}\ \bar{v}_x \right)_i^{j} + \Delta x_i \left(\bar{W}_f \bar{B} \right)_i^{j} \Bigg] = 0 \end{aligned} \qquad (9)$$

in which

$$\overline{A}_i = \frac{A_i + A_{i+1}}{2} \tag{10}$$

$$\overline{S}_{f_i} = \frac{\overline{n}_i^2 \left|\overline{Q}_i\right| \overline{Q}_i}{2.208\ \overline{A}_i^2 \overline{R}_i^{4/3}} \tag{11}$$

$$\overline{B}_i = \frac{B_i + B_{i+1}}{2} \tag{12}$$

$$\overline{Q}_i = \frac{Q_i + Q_{i+1}}{2} \tag{13}$$

$$\overline{R}_i = \overline{A}_i / \overline{B}_i \tag{14}$$

$$\overline{S}_{e_i} = \frac{K_{e_i}}{2g\Delta x_i} \left| \left(\frac{Q}{A}\right)^2_{i+1} - \left(\frac{Q}{A}\right)^2_i \right| \tag{15}$$

$$\overline{W}_{f_i} = C_{w_i} \left|\overline{V}_{r_i}\right| \overline{V}_{r_i} \tag{16}$$

$$\overline{V}_{r_i} = \frac{\overline{Q}_i}{\overline{A}_i} - \overline{V}_{w_i} \cos\omega \tag{17}$$

where V_r is the velocity of the wind relative to the velocity of the flow in the channel and ω is the acute angle the wind makes to the channel. The bar(-) above the variables represents the average of that particular variable over the reach length (Δx_i) between the net points i and i+1. The subscript (i) associated with $\overline{q}$, $\overline{v}_x$, $\overline{A}$, $\overline{B}$, $\overline{S}_f$, $\overline{Q}$, $\overline{S}_e$, $\overline{W}_f$, $\overline{V}_r$ and $\overline{V_w}$ represents the number of the reach, (Δx_i) rather than the node number. Node numbers commence with 1 and terminate with N, while reach numbers commence with 1 and terminate with (N-1).

Equations 8 and 9 are nonlinear with respect to the unknowns h and Q at the net points on the j+1 time line. All terms associated with the j^{th} time line are known from either the initial conditions or previous computations. The initial conditions are values of h and Q at each computational point (node) along the x-axis for the first time line (j+1).

They are obtained from a previous unsteady flow solution, a steady flow backwater solution or they can be estimated since small errors in the initial conditions dampen out within a few time steps.

Equations 8 and 9 are two nonlinear algebraic equations which cannot be solved in a direct (explicit) manner since there are four unknowns, h and Q at points i and (i+1) on the (j+1) time line and only two equations. However, if similar equations are formed for each of the (N-1) Δx reaches between the upstream and downstream boundaries, a total of (2N-2) equations with 2N unknowns results. (N denotes the total number of computational points or cross sections.) Then prescribed boundary conditions, one at the upstream extremity of the river and one at the downstream extremity, provide the necessary two additional equations required for the system to be determinate. The resulting system of 2N nonlinear equations with 2N unknowns is solved by a functional iterative procedure, the Newton-Raphson method (Amein and Fang, 1970).

In the iterative procedure, trial values using linear or parabolic extrapolation from solutions of h and Q at previous time steps are assigned to the 2N unknowns. Substitution of these into the system of 2N nonlinear equations yields a set of 2N residuals. The Newton-Raphson method seeks to reduce the residuals to an acceptable tolerance level which is usually achieved within one or two iterations.

In the Newton-Raphson method, a system of 2N x 2N linear equations are generated. The coefficient matrix of the system is composed of partial derivatives which are functions of the unknowns; however, the elements in the coefficient matrix can be assigned numerical values by substituting the trial values for the unknowns. The coefficient matrix is related to the set of 2N residuals by a set of 2N corrections to the original trial values of the unknowns. It is the 2N corrections that are sought in the solution of the 2N x 2N linear system. The coefficient matrix has a banded structure with at most four elements in any row. This property allows the use of a special modified Gaussian elimination algorithm for solving the system (Fread, 1971). Modification of the elimination algorithm reduces the core storage from $4N^2$ to 8N and the number of computational operations are reduced from the order of $(16/3N^3+8N^2)$ to 38N. The increase in computational efficiency is critical to the feasibility of the implicit solution technique.

Automatic Fixup Procedure

When nonconvergence occurs as the result of the water surface elevation going below the minimum elevation describing the cross section or when the maximum number of iterations for the Newton-Raphson procedure have been exceeded, DWOPER has an automatic fixup procedure to help correct the problem and continue routing. The following steps are taken when satisfactory answers are not obtained during routing:

1. Reduce the computational time step by ½ and repeat the solution procedure.
2. Reduce the time step again by ½ and repeat the solution procedure.
3. Reduce the time step again by ½ ; increase the Θ weighting factor by 0.05 and repeat the solution procedure.
4. Keep the time step the same as it was in the previous step; increase the Θ weighting factor by 0.05; and repeat the solution procedure.
5. Repeat the previous step until the theta weighting factor exceeds 1.00.
6. Go back to the original time step and linearly extrapolate.

To help prevent nonconvergence due to low flow conditions, a minimum value for stage or discharge (depending on the boundary condition) is read in for each hydrograph at the upstream boundary. If at any time during simulation, the upstream boundary goes below this minimum value, the minimum value is used for the current time step only. When the actual value of the upstream hydrograph goes above the minimum value, the actual value is used. The minimum value at each upstream boundary must be calibrated. If the inflow hydrograph is generated using a mathematical function, then the minimum value is set equal to the initial stage or discharge at the upstream boundary.

The program will linearly extrapolate up to a total of eight times. If no satisfactory solution has been reached, then the water surface elevations and discharges for future time steps will be set equal to the last extrapolated values.

Initial Conditions

DWOPER allows initial conditions to be obtained from the following sources:

1. Estimated stages and discharges at each cross section are read in;
2. Observed stages at each cross section where a river gage is located are read in; stages at intermediate cross sections are linearly interpolated within the model; observed discharges at the upstream extremity of the main stem river and each tributary are also read in; all downstream discharges are determined by the summation of flows from the upstream to downstream boundaries including tributary inflow to the main stem and lateral inflow occurring along either the main stem or tributaries;
3. Computed stages and discharges which have been saved from a previous unsteady flow simulation; and
4. Assumed steady flow to obtain discharges and followed by a back-water computation to obtain stages.

In each case, the unsteady flow equations are solved for several time steps using the initial conditions together with boundary conditions which are held constant during the time steps. This allows the errors in the initial conditions to dampen out which results in the initial conditions being more nearly error free when the actual simulation commences and transient boundary conditions are used. This warm-up is not used when forecasting in which previously stored carryover is used as the initial conditions.

Boundary Conditions

Boundary conditions must be specified in order to obtain solutions to the St. Venant equations. In fact, in most unsteady flow problems, the unsteady disturbance is introduced into the flow at the boundaries or extremities of the river system. DWOPER can readily accommodate either of the following boundary conditions at the upstream extremities of the river system:

1. known stage (water surface elevation) hydrograph, $h_1(t)$; or
2. known discharge hydrograph, $Q_1(t)$.

Downstream boundary conditions included as options in DWOPER are:

0. known tide hydrograph $h_n^{(t)}$
1. known stage (water surface elevation) hydrograph, $h_N(t)$;
2. known discharge hydrograph, $Q_N(t)$; or
3 a known relationship between stage and discharge such as a Rating Curve.

With respect to the Rating Curve boundary condition, the rating may be single-valued and read in as tabular (piece-wise linear) values of stage and discharge with linear interpolation provided internally for intermediate values. The rating may also be a loop Rating Curve generated internally from cross section and roughness properties of the downstream extremity and the instantaneous water surface slope at the instantaneous water surface slope at the previous time step.

Cross Sections

Irregular as well as regular geometrical shaped cross sections are acceptable in DWOPER. Each cross section is read in as a piece-wise linear relationship. Experience has shown that in almost all instances the cross section may be sufficiently described with eight or less sets of widths and associated elevations. A low-flow cross-sectional area which can be zero is used to describe the cross section below the minimum elevation read in. From this input, the cross-sectional area associated with each of the widths is initially computed within the model. During the solution of the unsteady flow equations, any areas or widths associated with a particular water surface elevation are linearly interpolated from the piece-wise linear relationships of width to elevation read in or the area-elevation sets initially generated within the model. If the water surface elevation exceeds the maximum elevation in the table, the model will linearly extrapolate beyond the last 2 valid points. If the water surface elevation is less than the minimum elevation in the table, the model will linearly extrapolate beyond the first two valid points in the table.

Cross sections at gaging station locations are always used as computational points in the x-t plane. Cross sections are also specified at points along the river where significant cross-sectional changes occur or at points where major tributaries enter. Typically, cross sections for large rivers (e.g. Mississippi, Ohio) with slowly varying transients may be spaced as much as 5-20 miles apart. The maximum distance between cross sections is dependent on the channel properties, duration and rate of rise of the wave and the computational step sizes of Δx and Δt. The relation of each to the accuracy of the simulation is defined by Fread (1974). Generally, the cross section spacing Δx should be equal to the wave speed (c) multiplied by the Δt time step, e.g., Δx (mi) = c (mi/hr) * Δt (hr). Usually, 'average' cross sections are placed midway between cross sections associated with gaging stations or significant geometry changes. The average cross section is obtained from a special data processing program in which cross sections at all sections such as crossings, bends or other changes in cross section geometry which violate the assumption of linear variation between adjacent cross sections are averaged together to obtain a weighted average. The distance between each cross section is used as the weighting factor.

Off-Channel Storage

Dead storage areas wherein the flow velocity in the x-direction is considered negligible relative to the velocity in the active area of the cross section is a feature of DWOPER. Such dead or off-channel storage areas can be used to effectively account for embayments, ravines or tributaries which connect to the flow channel but do not pass flow and serve only to store the flow. Another effective use of off-channel storage is to model a heavily wooded flood-plain which stores a portion of the flood waters passing through the channel. In each of these cases, the use of zero velocity for the portion of flood waters contained in the dead storage areas results in a more realistic simulation of the actual flow than using an average velocity derived from the main flow channel and the dead storage area. The off-channel storage cross-sectional properties are described in the same way as the active cross-sectional areas, i.e., for each section, a table of topwidths and elevations is read in along with the area associated with the lowest elevation. A table of area-elevation is created within DWOPER and intermediate storage topwidths or areas are linearly interpolated from the two tables as required.

Roughness Coefficients

Manning's n is used to describe the resistance to flow due to channel roughness caused by bed forms, bank vegetation and obstructions, bend effects and eddy losses. Manning's n is defined for each channel reach bounded by gaging stations and is specified as a function of either stage or discharge according to a piece-wise linear relation with both n and the independent variable (h or Q) read in to DWOPER in tabular form. Linear interpolation is used to obtain n for values of h or Q intermediate to the tabular values. If the computed stage or discharge goes beyond the range of the table values, then the model uses the extreme values of the table.

The computations are often sensitive to Manning's n. Although in the absence of necessary data (observed stages and discharges), n can be estimated; however, best results are obtained when n is adjusted to reproduce historical observations of stage and discharge. The adjustment process is referred to as calibration. This may be either a trial-error process or an automatic iterative procedure available within DWOPER. The automatic calibration feature is described later.

Lateral Inflows

DWOPER incorporates tributary inflows using the lateral inflow term, q, in Equations 1 and 2. The inflows are considered to be independent of flows occurring in the river to which they are added. They are read in as a time series of flows with a constant time interval. They may be described for any Δx reach along the river. The flow is specified in cfs; it is the sum of all lateral inflows occurring within a particular Δx reach. Outflows may also be simulated by assigning a negative sign to the flow value in the time series. Linear interpolation is used to provide flow values at times other than those of the time series which are determined by the time intervals associated with the flows.

Local Losses

The effects of local head losses incurred at severe contractions and/or expansions such as bridge openings are accounted for by the term S_e as defined by Equation 4. This necessitates an expansion-contraction coefficient associated with each Δx reach to be read in. The local head loss is in addition to the head losses incurred by the flow resistance due to channel roughness associated with the Manning's n.

Wind Effects

The effect of wind resistance on the surface of the flow is accounted

$$W_f = C_w \left(V_r \cos \omega \right)^2 \tag{18}$$

for by the term W_f in Equation 2. W_f is defined as:
in which V_r is the velocity of the wind relative to the velocity of the channel flow, ω is the angle between the wind direction and

channel flow direction and C_w is the non-dimensional wind coefficient. C_w may be estimated from empirical studies or it may be assigned a value using a trial-error calibration process.

Lock and Dam Condition

A river system may include small dams with gates to pass the river flow in such a way as to maintain certain water surface elevations on the upstream side of the dam. Usually associated with the dam is a lock for allowing navigation of river craft and barges past the dam. DWOPER can accommodate any number of lock and dam installations within the river system being simulated. A 'through' computation scheme is used as opposed to separating the river system into discrete portions because of the lock and dam and specifying external boundary conditions applicable to the lock and dam. The through computation scheme allows the simultaneous simulation of the entire river system including portions with lock and dams. This facilitates data preparation and allows a correct simulation of backwater effects when the tailwater elevation below the dam raises to an elevation such that the pool elevation on the upstream side is no longer controlled by operation of the gates.

In the through computation scheme a specified critical tailwater elevation which is read in for a particular lock and dam is used to determine if the pool elevation is controlled by the gate operation or by channel flow. If the simulated tailwater is less than the critical elevation, the conservation of mass equation, Equation 8, is replaced by and the conservation of momentum equation, Equation 9, is replaced by:

$$Q_i^{j+1} - Q_{i+1}^{j+1} = 0 \tag{19}$$

and the conservation of momentum equation, Equation 9, is replaced by

$$h_i^{j+1} - h_t = 0 \tag{20}$$

where h_t is the target pool elevation which the dam tender attempts to maintain using operation of the gate. The target pool elevation may be a constant value or it may be specified as a function of time and read in as a time series. It may also be determined from a Rating Curve as a function of discharge.

When the simulated tailwater elevation exceeds a critical tailwater elevation, the flow passes through the dam according to the Equations 1 and 2 as in any other Δx reach.

Flow Diversions

Special flow diversions through diversion control structures are accommodated within DWOPER. The user specifies the percent of flow upstream of the diversion structure to be diverted at each time step and that quantity of flow is removed from the channel.

Dendritic River Systems

Although the implicit formulation of the unsteady flow equations is well suited for simulating unsteady flows in a system of rivers - in that the response of the system as a whole is determined during each time step - particular care must be given to maintain the necessary solution efficiency as mention previously with regards to the matrix solution technique of the Newton-Raphson procedure. An efficient solution technique for dendritic (tree-type) river systems is utilized in DWOPER. This technique, as described by Fread (1973), solves during a time step the unsteady flow equations first for the main stem and then for each tributary of the river system. The tributary flow at the confluence of the tributary and main stem river is treated as lateral flow q which is first estimated when solving the equations for the main stem.

The tributary flow depends on its upstream boundary condition, lateral inflows along its reach and the water surface elevation at the confluence which is obtained during the simulation of the main stem. Due to the interdependence of the flows in the main stem and its tributaries, an iterative or relaxation procedure is necessary. Convergence is attained when the estimated tributary flow at the confluence is sufficiently close to the computed flow for the tributary at its downstream boundary using the main stem water surface elevation for the downstream boundary condition. Usually one or two iterations is sufficient for convergence to a suitable tolerance.

DWOPER can accommodate any number of tributaries. Although the iterative algorithm is designed for first order tributaries, systems with second order tributaries may sometimes be accommodated by reordering the system, i.e., selecting another branch of the system as the main stem.

Weir-Flow Bifurcations

In DWOPER, any number of Δx reaches along the channel may bypass flow to a fictitious channel which connects back into the main channel at some point downstream from the bifurcation. The flow in the bypass channel which may affect the weir flow is accounted for by a submergence correction to the weir flow. The crest elevation of the overbank section which acts as the weir-flow bypass is specified. Each section has a discharge coefficient which may be estimated or obtained through trial-error calibration. The location along the channel where the bifurcation(s) occur, the average crest elevation of each such Δx reach and the discharge coefficient are read in as input data. Levee overtopping and/or failure can be simulated using the weir flow bifurcation feature.

There are some limitations to the weir-flow bifurcation option. Weir flow can only be passed from the main channel. Levee overtopping cannot be simulated on tributaries. Also, since DWOPER cannot handle a dry channel, the fictitious tributary receiving flow from the main channel must have a small base flow. Because the program knows that this is an artificial channel, the small base flow will not be added back into the main channel. The actual weir flow will, however, be passed back into the main channel unless otherwise specified by the user.

Computer Core and Computational Requirements

DWOPER has been created using the programming feature, 'variable dimensioning'. This enables the size of the arrays of subscripted variables such as observed hydrograph stages, cross section topwidths, computed stages and discharges, etc., to be changed from one simulation run to the next. There is a maximum total size for the sum of all arrays, but within that bound the allocation of core space among the variables is flexible. The individual array space is allocated based on data input parameters (no. of rivers, maximum no. of cross sections on any river, no. of topwidths used to describe a cross section, etc.). From these parameters, the maximum amount of space needed for each array can be specified and retained in the parameter array. This flexibility allows maximum use of the core set apart for DWOPER.

The implicit formulation of the basic dynamic wave computational element allows the time step size to be selected according to accuracy requirements rather than numerical stability considerations. This factor makes DWOPER very efficient in the use of computer time. Computational requirements are approximately 0.004 sec/time step/distance step (IBM 360/195 computer).

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