Introduction

This Section describes the methods available for converting a given
stage or discharge to the associated discharge or stage using
interpolation or extrapolation of a loop Rating Curve.

A loop Rating Curve is one that has a different stage-discharge
relationship when the river level is rising than when it is falling
as illustrated in Figure 1.

The loop-rating stage-discharge conversion method was originally
developed by Fread (1973, 1975) for simulating the dynamic
relationship that exists between stage and discharge when the energy
slope is variable due to the effects of variable water surface slope
and flow accelerations of unsteady, non-uniform flow. These effects
caused by changing discharge can produce a 'loop' in the stage-
discharge Rating Curve such that two different stages exist for each
discharge. The lesser stage is associated with the rising limb of
the discharge hydrograph while the greater stage occurs during
recession of the floodwave. Because this loop is due to the changing
or dynamic nature of the floodwave it is termed a 'dynamic loop'.

A unique dynamic stage-discharge relation for a particular location
along a channel can be determined via a mathematical model based on
the complete one-dimensional equations of unsteady flow and the
Manning equation which accounts for energy losses due to the
resistance of the channel boundary. These equations are derived in
several references (Chow (1959) and Henderson (1966)) and are:

$$A \; \frac{\partial V}{\partial x} \; + V \; \frac{\partial A}{\partial x} \; + B \; \frac{\partial h}{\partial t} \; = 0 \tag{1}$$

$$\frac{\partial V}{\partial t} \; + V \; \frac{\partial V}{\partial x} \; + g \left(\frac{\partial y}{\partial x} + S - S_0 \right) \; = 0 \tag{2}$$

and

$$Q = \frac{\mu}{n} \; A \; R^{2/3} \; S^{1/2} \tag{3}$$

where x   is the distance along the channel in units of FT
      t   is the time in units of SEC
      A   is the channel cross-sectional area in units of FT2
      h   is the water surface elevation above a datum plane in
            units of FT
      y   is the depth of flow in units of FT
      S   is the energy slope in units of FT/FT
      V   is the mean velocity of flow across the section in units
            of FT/SEC
      S_0 is the effective bottom slope of the channel in units of

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            FT/FT
      g   is the acceleration due to gravity in units of 32.2
            FT/SEC$^2$
      n   is the Manning's coefficient in units of SEC/FT$^{1/3}$
      Q   is the discharge in units of FT3/SEC
      R   is the hydraulic radius in units of FT
      μ   is 1.49 if English units or 1.0 if Metric units
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In the development that follows the following assumptions are made
for a short section of channel containing the gaging station or
forecast point:

1. Lateral inflow or outflow is negligible;
2. The channel width is essentially constant; i.e. $\partial B/\partial x = 0$;
3. Energy losses from channel friction and turbulence are
   described by the Manning equation;
4. The geometry of the section is essentially permanent; i.e. any
   scour or fill is negligible;
5. The bulk of the flood wave is moving approximately as a
   kinematic wave which implies that the energy slope is
   approximately equal to the channel bottom slope; and
6. The flow at the section is controlled by the channel geometry,
   friction and bottom slope and by the shape of the flood wave.

An expression for the energy slope is obtained by rearranging
Equation 2 in the following form:

$$S = S_0 - \frac{\partial y}{\partial x} - \frac{V}{g}\frac{\partial V}{\partial x} - \frac{1}{g}\frac{\partial V}{\partial t} \tag{4}$$

The four terms on the right side of Equation 4 represent the
component slopes which produce the variable energy slope S due to
changing discharge. From left to right respectively the four slopes
are attributed to: gravity force, pressure force, convective
(spatial) acceleration and local (temporal) acceleration.

Using assumption (2) Equation 1 may be expressed as:

$$\frac{\partial V}{\partial x} = - \frac{VB}{A}\frac{\partial y}{\partial x} - \frac{B}{A}\frac{\partial h}{\partial t} \tag{5}$$

For river valleys with wide, flat flood plains (with width B_f), the
flow velocity of water overtopping the channel bank (with width B_m) is
reduced as the debris and vegetation encountered in the flood plain
resists flow and greatly increases the composite roughness
coefficient. As the flow velocity in the overbank area (with width
B_o) is nearly negligible near the flood plain ground surface, the area
may be considered as off-channel storage as shown in Figure 2.

The composite roughness coefficient is estimated from:

$$n = \left(\frac{n_m^2 \, B_m + n_f^2 \, B_f}{B_c} \right)^{1/2}$$

where n_m is the roughness coefficient for channel
n_f is the roughness coefficient for flood plain
B_c is $B_m + B_f$

This overbank storage area must still be accounted for in the continuity equation while, because of negligible flow through it, it may be neglected in the momentum equation. The expression for topwidth, B, in Equation 5 is thus composed of two components: the active flow area topwidth, B_c and the inactive, off-channel flow area topwidth, B_0. Equation 5 may now be written as:

$$\frac{\partial V}{\partial x} = - \frac{VB_c}{A} \frac{\partial y}{\partial x} - \frac{B_c}{A} \frac{\partial h}{\partial t} - \frac{B_0}{A} \frac{\partial h}{\partial t} \qquad (5a)$$

Upon substituting Equation 5a and V=Q/A in Equation 4 the following expression for the variable energy slope S is obtained:

$$S = S_o + \left(\frac{Q2 \, B_c}{g \, A^3} - 1 \right) \frac{\partial y}{\partial x} + \left(\frac{B_c + B_o}{g \, A^2} \right) Q - \frac{\partial h}{\partial t} - \frac{1}{g} \frac{\partial (Q/A)}{\partial t} \qquad (6)$$

Information concerning the characteristics of either the stage or discharge hydrograph generally is available only at the location for which the Rating Curve is required. Such lack of spatial resolution of the hydrograph requires that the derivative terms with respect to x in Equations 4 and 5 be replaced by equivalent expressions which can be evaluated from the available information. Henderson (1966) shows that if the bulk of the flood wave is moving approximately as a kinematic wave then the following expression may be used to eliminate the need for spatial resolution of the specified hydrograph:

$$\frac{\partial y}{\partial x} = - \frac{1}{c} \frac{\partial h}{\partial t} - \frac{2 \, S_o}{3 \, r^2} \qquad (7)$$

where c is the kinematic wave velocity
r is the ratio of the channel bottom slope to an average wave slope

The kinematic wave velocity c may be determined from observations of the time interval between equal rises in stage, h, at gaging stations along the channel. Also, c may be computed from a relationship given by Henderson (1966) and Chow (1959):

$$c = \frac{1}{B_c} \frac{dQ}{dh} \qquad (8)$$

where dQ/dh is the slope of the single-value Rating Curve

If the channel is assumed to be prismatic then the kinematic wave

velocity can be computed directly by substituting Equation 3 in Equation 8. After differentiation then the following is obtained:

$$c = KV = KQ/A \tag{9}$$

where

$$K = \frac{5}{3} - \frac{2A}{3B_c^2} \, dB_c/dh \tag{10}$$

and the hydraulic radius R is approximated by the hydraulic depth D as follows:

$$R = D = A/B_c \tag{11}$$

This is a good approximation of the hydraulic radius for large channels. If the hydraulic radius is used in lieu of the hydraulic depth then the term dB_c/dh in Equation 10 would be replaced by dP/dh, where P is the wetted perimeter of the channel cross-section and the term B_c2 would be replaced by the product PB_c. From an inspection of Equation 10 it is evident that K has an upper limit of about 1.7 when dB/dh is negligible and a lower limit of about 1.3 for a triangular-shaped channel. It has been observed that K can be approximated as 1.3 for many natural channels (Corbett, 1943; Linsley, et al., 1949). Although there are a number of methods for determining the kinematic wave velocity, Equation 9 is used in this study.

$$c_d = V \pm (g \, A/B_c)^{1/2} \tag{12}$$

It should be noted that the wave velocity is frequently greater than that computed by Equation 9; however, this may result from the fact that the wave is more nearly a dynamic wave than a kinematic wave. The dynamic wave velocity as given by Henderson (1966) is:A comparison of Equation 9 with Equation 12 indicates that the dynamic wave velocity can be considerably larger than the kinematic wave velocity, particularly for large A/B ratios. The dynamic wave predominates over the kinematic wave when the channel flow is pooled such as behind a dam or other constriction in the channel. Under this condition the flow is not controlled by channel geometry, friction, bottom slope and the shape of the flood wave; therefore, flow in pooled areas where dynamic waves are formed is not treated herein. In some instances when pooling occurs only during the lower stages the wave velocity changes from that of a dynamic wave to more nearly that of a kinematic wave as the stage increases. The portion of the Rating Curve, associated with the higher stages when the kinematic wave approximation is more applicable, could be determined approximately by the method developed herein.

The value of r in Equation 7 may be taken as a constant for a particular channel. Typical values of r range from 10 to 100. It is used in Equation 7 as part of a small correction which accounts for the fact that a typical flood wave is not exactly a kinematic wave. To arrive at a value for r the wave slope is approximated from the characteristics of a typical flood event for a particular channel location. The wave slope is determined by dividing the height of the

wave by its half-length, the latter obtained by assuming that the
wave travels as a kinematic wave during the interval of time from the
initiation of the wave to the occurrence of the wave peak at the
location of concern. The half-length is determined from the product
of the average kinematic wave velocity and the time to peak stage.
Equation 9 is used to determine the average kinematic velocity, with
Q and A taken as the average values during the flood event and K
assumed equal to 1.3. Therefore the following expression is obtained
for evaluating r:

$$r = \frac{56200 \left(Q_p + Q_o\right) T\ S_o}{\left(h_p - h_o\right) A} \tag{13}$$

> where Q_o is the discharge at beginning of typical flood in units
>   of CFS
> Q_p is the peak discharge for typical flood in units of CFS
> h_o is the stage at beginning of typical flood in units of FT
> h_p is the peak stage of typical flood in units of FT
> A   is the cross-sectional area associated with the average
>   stage, $(h_p + h_o)/2$ in units of FT2
> T   is the interval of time from beginning of rise in stage
>   until the occurrence of the peak stage in units of
>   days

Since c and r are defined by Equations 9 and 13, Equation 7 can be
substituted in Equation 6, with the partial derivatives in the latter
replaced by finite difference notation. After some rearrangement the
following equation is obtained:

$$S = S_o + \left[\frac{A}{KQ} + \left(\left(1 - \frac{1}{k}\right) B_c + B_o \right) \frac{Q}{gA^2} \right] \delta h_s$$

$$+ \frac{Q'/A' - Q/A}{g\Delta t} + \frac{2^{S_o}}{3r^2} \left(1 - \frac{B_c Q^2}{gA^3} \right) \tag{14}$$

> where Δt is the data time interval in units of SEC
> Q' is the discharge at time $t - \Delta t$ in units of CMS
> A' is the cross-sectional area at time $t - \Delta t$ in units of M2
> δh_s is the change in water surface elevation during the time
>   interval Δt in units of FT/S

$$h_s = \frac{h - h'}{\Delta t}$$

> where h'   is the stage at time $t - \Delta t$

Equation 14 is the expression for the variable energy slope S which
is caused by varying discharge. All the terms on the right side of
the equation except S_o account for the effect of the dynamic
characteristic of the flow. If the flow is steady (unchanging with
time) the energy slope is constant and equivalent to the bottom
slope, S_o. This is evident from Equation 14 since all terms on the

right side of the equation except the first term vanish when the flow is steady; i.e. $Q' = Q$, $\delta h_s = 0$ and r is infinitely large since the wave slope vanishes for steady uniform flow.

An expression may be derived for the dynamic relation between stage and discharge when the energy slope is variable due to changing discharge. This can be obtained by substituting the hydraulic depth for the hydraulic radius in Equation 3 and using Equation 14 for the variable energy slope:

$$
Q - 1.0 \frac{AD^{2/3}}{n} \left[S_o + \left[\frac{A}{KQ} + \left(\left(1 - \frac{1}{K} \right) B_c + B_o \right) \frac{Q}{gA^2} \right] \delta h_s \right.
$$

$$
\left. + \frac{Q'/A' - Q/A}{g\Delta t} + \frac{2S_o}{3r^2} \left(1 - \frac{B_c Q2}{gA^3} \right) \right]^{1/2} = 0
$$

(15)

This equation differs from others given in the literature (e.g., Linsley, et al. (1949), Corbett (1943)) which include only terms equivalent to the first two terms of the variable energy slope of Equation 4.

Equation 15 forms the basis of a model that can be used to determine either discharge when the rate of change of stage is known (as in stream gaging) or stage when the rate of change of discharge is known (as in stream forecasting). These alternative conditions are denoted respectively as Case A and Case B. A brief description of each follows:

   Case A: The discharge hydrograph is determined from a specified
           (observed) stage hydrograph. In this case, Q is unknown
           in Equation 15; the known quantities consist of constants
           (S_o, r, g, Δt), known functions (A, B, n, K) of the
           specified stage h and known quantities (Q', A', h')
           associated with time $t-\Delta t$.

   Case B: The stage hydrograph is determined from a specified
           (predicted) discharge hydrograph. In this situation, h is
           the unknown in Equation 15 and the terms (A, B, n, K) are
           functions of h. The known quantities consist of constants
           (S_o, r, g, Δt), the specified discharge Q and the
           quantities (Q', A', h') associated with time $t-\Delta t$.

The unknowns Q and h in Case A and Case B respectively are not expressed in an explicit manner and require an iterative solution. An efficient method for obtaining a solution is by using Newton Iteration (Issacson and Keller, 1966).

Solution by Newton Iteration

A nonlinear equation may be solved by a functional iterative technique such as Newton iteration. Consider the following equation expressed in functional form:

$$f(x) = 0 \qquad\qquad (NR-1)$$

The solution of Equation NR-1 is obtained in an iterative manner, proceeding from a first solution estimate x^k towards succeeding improved estimates x^{k+1}, which tend to converge toward an acceptable solution. The orderly procedure by which the improved solution estimate x^{k+1} is obtained so that it converges to an acceptable solution is known as Newton iteration and is described as follows.

A nonlinear equation such as Equation NR-1 may be linearized by using only the first two terms of its Taylor series expansion at x^k:

$$f(x) = f(x^k) + \frac{df(x^k)}{dx^k} (x-x^k) \qquad\qquad (NR-2)$$

The right side of Equation NR2 is the linear function of x^k that best approximates the nonlinear function $f(x)$ which is evaluated at x^k. An iterative procedure, which will cause $f(x^k)$ to approach zero as the quantity $(x-x^k)$ approaches zero, can be obtained from Equation NR-2 by setting $f(x)$ equal to zero and replacing x with x^{k+1}, which will be an improved solution estimate for x if the iterative procedure is convergent. Therefore Equation NR-2 takes the form:

$$x^{k+1} = x^k - \frac{f(x^k)}{df(x^k)/dx^k} \qquad\qquad (NR-3)$$

    where k is the iteration number

Equation NR3, the general iteration algorithm of Newton, is repeated until the difference $(x^{k+1} - x^k)$ is less than ϵ which is a suitable error tolerance for the solution of Equation NR-1. When this occurs the iteration process has converged; i.e. x^{k+1} has approached x to within the prescribed error tolerance ϵ.

The convergence of the iteration process depends on a good first solution estimate $x^{k=1}$. If the estimate is sufficiently close to x, convergence is attained and it is at a quadratic rate, i.e. second order, since the iterative procedure involves the first derivative. The nonlinear equation which is solved by the Newton iterative algorithm in this report is a time dependent finite-difference equation. A first estimate of the solution is obtained by using the solution associated with the time t-Δt. In this study the iteration process always converged. The convergence process can be hastened when the first solution estimate $x^{k=1}$ is made closer to the acceptable solution. A simple linear extrapolation is used to provide better first solution estimates. Thus:

$$x_j^{k=1} = x_{j-1} + (x_{j-1}-x_{j-2})/2 \qquad\qquad (NR-4)$$

    where j   is the solution at time t
           j-1 is the solution at time t-Δt, etc.

In the following equation the Newton iteration algorithm is applied to Equation 15, which is presented here for convenient reference:

$$Q - 1.0 \frac{A\, D^{2/3}}{n} \left[S_o + \left[\frac{A}{K\,Q} + \left(B_c + B_o - \frac{B}{K^c} \right) \frac{Q}{gA^2} \right] \delta h_s + \frac{Q'/A - Q/A}{G\Delta t} \right.$$

$$\left. + \frac{2S_o}{3r^2} \left(1 - \frac{B_c Q^2}{gA^3} \right) \right]^{1/2} = 0 \qquad\qquad \text{(NR-5)}$$

First the Case A condition is treated where the discharge Q is the unknown in Equation NR-5 and then the Case B condition is presented in which the stage h is the unknown.

Case A

Equation NR-5 can be solved for Q at time t as follows:

$$Q^{k+1} = Q^k - \frac{f(Q^k)}{df(Q^k)/dQ^k} \qquad\qquad \text{(NR-6)}$$

where   k   is the iteration number
      $f(Q^k)$ is Equation NR-5 evaluated with the unknown Q
                   replaced by the approximation Q^k
      $df(Q^k)/dQ^k$ is the derivative of $f(Q^k)$ with respect to Q^k

Thus:

$$f(Q^k) = Q^k - L_2 L_o^{1/2} \qquad\qquad \text{(NR-7)}$$

where

$$L_o = L_3 + \frac{L_4}{Q^k} + L_5 Q^k + L_6 (Q^k)^2 \qquad\qquad \text{(NR-8)}$$

$$L_2 = 1.0 \frac{AD^{2/3}}{n} \qquad\qquad \text{(NR-9)}$$

$$L_3 = S_o + \frac{2S_o}{3r^2} + \frac{Q'}{g\,A'\Delta t'} \qquad\qquad \text{(NR-10)}$$

$$L_4 = \frac{A\,\Delta h_s}{K} \qquad\qquad \text{(NR-11)}$$

$$L_5 = \left(B_c + B_o - \frac{B}{K^c} \right) \frac{\Delta h_s}{gA^2} - \frac{1}{G\,A\,\Delta t} \qquad\qquad \text{(NR-12)}$$

$$L_6 = - \frac{2 \ S \ B_c}{3 \ r^2 \ gA^3} \tag{NR-13}$$

in which

$$D = \frac{A}{B_c} \tag{NR-14}$$

$$\delta h_s = \frac{(h - h')}{\Delta t} \tag{NR-15}$$

$$K = \frac{5}{3} - \frac{2}{3} \ \frac{A \ dB_c \ /dh}{B^2} \tag{NR-16}$$

$$dB_c/dh = \frac{\left(B_c - B_c'\right)}{h - h'} \tag{NR-17}$$

$$n = \text{Manning's n at } h^k \tag{NR-18}$$

Also:

$$\frac{df(Q^k)}{dQ^k} = 1 - 0.5 \ \frac{L_2 L_1}{L_o^{1/2}} \tag{NR-19}$$

where

$$L_1 = \frac{dL_o}{dQ^k} = - \frac{L_4}{(Q^k)^2} + L_5 + 2 \ L_6 \ Q^k \tag{NR-20}$$

In the above equations, A and B are known functions of the stage and are evaluated at h; B', A', h' are known from the time period previous; i.e. t' or t-Δt; S_o, r are constants.

Case B

Equation (NR-5) can be solved for h at time t as follows:

$$h^{k+1} = h^k - \frac{f(h^k)}{df(h^k)/dh^k} \tag{NR-21}$$

where k   is the number of iteration
      $f(h^k)$ is Equation NR-5 evaluated with the unknown h, which is also implicitly contained in the terms D, A, B, n, K and δh_s, replaced by the approximation h^k
      $df(h^k)/dh^k$ is the derivative of $f(h^k)$ with respect to h^k.

Thus:

$$f(h^k) = Q - 1.0^{2/3}\frac{AD^{1/2}J_o}{n}$$

where

$$J_o = J_2 + \left[J_3A + J_4\left[\frac{B_c+B_o-B_c/K}{A^2}\right]\right](h^k{-}h') + \frac{J_5}{A} + J_6\frac{B_c}{A^3}$$

in which

$$J_2 = S_o + \frac{2\,S_o}{3\,r^2} + \frac{Q'}{gA'\Delta t'}$$

$$J_3 = \frac{1}{K\,Q\,\Delta t'}$$

$$J_4 = \frac{Q}{g\,\Delta t}$$

$$J_5 = \frac{Q}{g\,\Delta t}$$

$$J_6 = \frac{-2\,S_o\,Q^2}{3_r 2\,g}$$

In the above:

$$K = \frac{5}{3} - \frac{2}{3}\,\frac{a}{B^2}\,dB_c/dh^k \qquad \text{(evaluated at } h^k)$$

$$dB_c/dh^k = \frac{(B_c^k - B_c')}{(h^k - h')}$$

and

$$\frac{df(h^k)}{dh^k} = -1.0\left[J_o^{1/2}\left[\frac{A}{n}\frac{d(D^{2/3})}{dh^k} + \frac{D^{2/3}\,B}{n} - \frac{A\,D^{2/3}\,dn/dh^k}{n^2}\right]\right.$$

$$\left. + \frac{0.5A\,D^{2/3}\,J_1}{n\,J_o^{1/2}}\right]$$

where

$$J_1 = \frac{d\,J_o}{d\,h^k} = J_3 A + J_4 \frac{(B_c + B_o - B_c/K)}{A^2}$$

$$+ (h^k - h') \left[J_3 B_c + \frac{J_4}{A^2} \left[dB_c + dB_o - \frac{dB_c}{K} \right) \right.$$

$$\left. - \frac{2B_c}{A} (B_c + B_o - B_c/K) \right.$$

$$\left. + \frac{B_c}{K^2} \frac{dK}{dh} \right] - \frac{A\,J_3}{K} \frac{dK}{dh} \right]$$

$$- \frac{J_5 B_c}{A^2} - \frac{J_6}{A^3} \left(\frac{dB_c}{dh} - \frac{3B_c}{D} \right)$$

(NR-32)

n = Manning's n at h^k (NR-33)

dn/dh^k = Change in Manning's n at h^{k-1}, h^k (NR-34)

$$\frac{d(D^{2/3})}{dh^k} = \frac{2}{3} D^{2/3} \left(\frac{B_c}{A} - \frac{dB_c/dh^k}{B_c} \right)$$

(NR-35)

In the above equations A and B are specified functions of the stage and are evaluated at h^k; B', A', h' are known from the previous time $t-\Delta t$ and g, Δt, S_o, r are constants.

For either Case A or Case B the solution of Equation NR-5 using Newton Iteration requires only about two iterations when the following convergence criteria are used:

$$|Q^{k+1} - Q^k| < \epsilon_Q$$

(NR-

and

$$|h^{k+1} - h^k| < \epsilon_h$$

(NR-

where

$$\epsilon_Q = 0.1 \text{ CMS}$$

(NR-38)

and

$$\epsilon_h = 0.001 \text{ M}$$

(NR-39)

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Figure 1. Stage-Discharge relationship for a loop Rating Curve
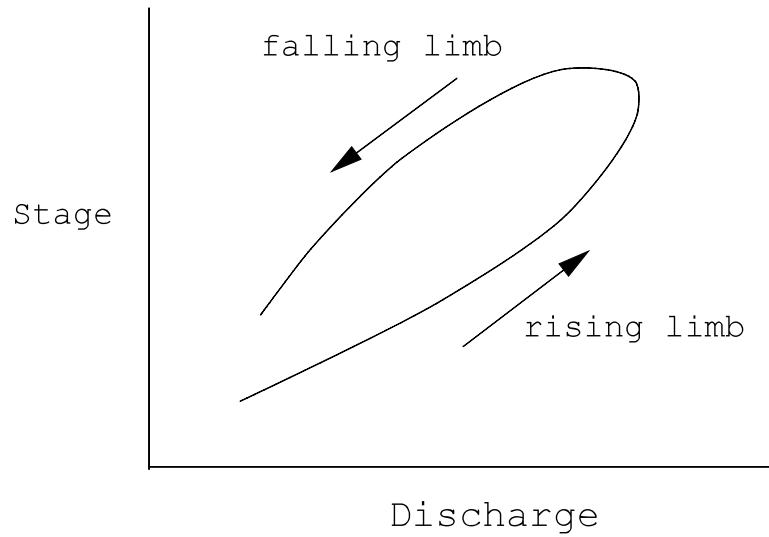

Figure 2. Typical cross-section showing active channel and off channel topwidth representation